International Journal of Economics and Financial Issues

ISSN: 2146-4138

available at http: www.econjournals.com

International Journal of Economics and Financial Issues, 2024, 14(1), 1-14.

Estimating Value at Risk and Expected Shortfall: A Kalman Filter Approach

Max van der Lecq[1], Gary van Vuuren[2]*

[1]School of Economics, University of Cape Town, Cape Town, South Africa, [2]Centre for Business Mathematics and Informatics, North-West University, Potchefstroom, 2351, South Africa. *Email: vvgary@hotmail.com

Received: 30 August 2023 **Accepted:** 04 December 2023 **DOI:** https://doi.org/10.32479/ijefi.15184

ABSTRACT

Value at Risk (VaR) estimates the maximum loss a portfolio may incur at a given confidence level over a specified time, while expected shortfall (ES) determines the probability weighted losses greater than VaR. VaR has recently been replaced by (but remains a crucial step in the computation of) ES by the Basel Committee on Banking Supervision (BCBS) as the primary metric for banks to forecast market risk and allocate the relevant amount of regulatory market risk capital. The aim of the study is to introduce a more accurate approach of measuring VaR and hence ES determined using loss forecast accuracy. VaR (hence ES) is unobservable and depends on subjective measures like volatility, more accurate (loss forecast) estimates of both are constantly sought. Modelling the volatility of asset returns as a stochastic process, so a Kalman filter (which distinguishes and isolates noise from data using Bayesian statistics and variance reduction) is used to estimate both market risk metrics. A variety of volatility estimates, including the Kalman filter's recursive approach, are used to measure VaR and ES. Loss forecast accuracy is then computed and compared. The Kalman filter produces the most accurate loss forecast estimates in periods of both calm and volatile markets. The Kalman filter provides the most accurate forecasts of future market risk losses compared with standard methods which results in more accurate provision of regulatory market risk capital.

Keywords: Kalman Filter, Value-at-Risk, Expected Shortfall
JEL Classifications: C3, C6, G21, G28

1. INTRODUCTION

Risk which arises from the activities of any given institution has historically required the use of metrics such as VaR and ES (or CVaR) to effectively identify and evaluate the potential losses a portfolio may face over a given time horizon and at a given confidence level. VaR defines the minimum expected loss for a given portfolio under normal market conditions, resulting in a frequency measure for losses beyond a certain confidence interval. Using a hypothetical daily portfolio VaR of $1 million at a 97.5% confidence level, there is a 97.5% chance that the portfolio will not exceed $1 million in losses for the specified day. Calculating the ES for the same portfolio requires a few additional steps to quantify the magnitude of losses beyond the VaR threshold. Put simply, it provides an average of the losses that exceed the loss level, at a prescribed confidence, determined by VaR (Acerbi and Tasche, 2002).

The application of accurate VaR estimates, particularly in banking institutions and investment firms, is of considerable importance. Capital allocation for market risk may be misaligned with the appropriate structures if the underlying risk is not adequately estimated, jeopardising the stability of the institutions that rely on these metrics. Several approaches, both non-parametric and parametric, have been established to better understand and manage the varying forms of risk (credit, operational, liquidity, and market) a firm may face (Manganelli and Engle, 2001). Economic reforms are often witnessed in a country, and markets can be susceptible to internal and external shocks, such as currency movements, credit rating changes, inflation, and shifts in risk premiums. Higher volatility during turbulent periods can result in financial returns having more distorted distributions than normal, making it difficult to assess VaR using standard methods (Miletic and Miletic, 2015). Krause and Tse (2016) highlight how recent empirical findings corroborate the earlier theoretical claims in existing literature that risk management leads to increased firm value and returns, while simultaneously decreasing return and cash flow volatility.

Although VaR has gained significant popularity in modern finance, it is plagued by some limitations and assumptions. ES has emerged as a preferred risk measurement tool in certain scenarios, offering advantages over (but still dependent upon) its progenitor. In 2013, the BCBS replaced VaR with ES as the new primary measure for banking institutions to forecast market risk (BCBS, 2013). The estimation of VaR and ES can be accomplished using various methods, each varying in popularity and complexity. In this article, a selection of these popular methods implemented in modern financial markets will be referred to and matched against the Kalman filter (Kalman, 1960), an algorithm which provides estimates of unknown variables and unobservable parameters through dynamic system estimation. While originally applied primarily in the field of engineering, the Kalman filter has more recently found applications in finance and economics. It has exhibited competence in estimating various factors, such as inflation expectations, commodity futures prices, and hedge ratios for interest rate contracts (Arnold et al., 2008). The objective is for the Kalman approach to serve as an alternative, and potentially more effective, method for financial analysts to quantify market risk.

The literature governing the application of the Kalman filter to financial risk is relatively scarce as the approach is still reasonably novel. This work is one of the first to provide robust, extensive results of comprehensive back-testing.

The remainder of this article proceeds as follows: Section 2 reviews the literature surrounding existing VaR and ES estimation methods, the Kalman filter, and its application in financial risk management. Section 3 sets out the underlying data and provides a summary of the relevant mathematics used to estimate VaR and ES as well as a detailed description of the workings of the Kalman filter. Section 4 presents and discusses the results of the subsequent analysis while Section 5 provides recommendations for further research and concludes.

2. LITERATURE REVIEW

Precise quantification of overall risk for a given institution or portfolio exposed to several systematic and unsystematic influences can be challenging. The coverage of risk detection in financial markets has historically focused on broad statistical concepts in standard deviation or variance. Since its introduction by 1994, VaR has undergone several adaptations, iterations, and additions in its rise to prominence as a suitable risk management tool. Its effectiveness has been validated by its ability to consolidate various com-ponents of market risk within a firm into a single quantitative measure. This attribute received substantial endorsement from industry and regulatory bodies, particularly in the late 1990s when the methodologies associated with the tool first became widely accessible (Marshall and Siegel, 1997).

Early contributions to VaR can be attributed to Markowitz (1952) and Roy (1952), who both emphasised the incorporation of covariances among risk factors to reflect diversification and hedging effects. However, due to limited processing power during subsequent decades, VaR remained primarily a theoretical concept. It was only when financial institutions began adopting VaR as a routine tool for assessing market risk and establishing risk limits that the Markowitz (1952) methodology gained widespread usage. In the late 1980s, JP Morgan developed RiskMetrics, a system capable of modelling numerous risk factors and employing various VaR metrics (JP Morgan, 1996). Prior to the introduction of VaR, commercial banks primarily focused on "desk by desk" risk assessment rather than considering overall company exposure (Chen, 2014).

Despite its widespread adoption in industry, VaR has been subject to scrutiny since its inception due to the identification of certain limitations and flaws. Artzner et al. (1999) and Acerbi and Tasche (2002) have previously questioned the viability of VaR, citing its lack of coherence as a risk measure. The primary argument was that there will consistently be a probabilistic chance of an extreme event taking place which falls a significant distance away from the estimate that VaR produces. This implied that VaR should not be relied on as a sole risk management tool.

VaR generally does not meet the subadditivity requirement, which states that a combination of the given risks for hypothetical assets A and B ultimately will not lead to an aggregate risk that is higher than the total of the individual risks (Daníelsson et al., 2012). This is not the case with VaR, ultimately discouraging di-versification. Practical implications of VaR not meeting the subadditivity requirement have been highlighted in the management of credit portfolio risk, for example. Credit instruments frequently feature "fat" tails in their return distributions and generally exhibit asymmetric return characteristics related to default risk-there may be a higher concentration of credit risk due to VaR's inability to meet the subadditivity requirement (Albanese, 1997).

In addition to subadditivity, three requirements for a coherent risk measure were defined by Artzner et al. (1999) as homogeneity, monotonicity, and translation invariance. Homogeneity implies a proportional level of risk relative to size, i.e., a twofold increase in an asset or a portfolio would equate to the same heightening in the level of risk. Monotonicity dictates that if a given portfolio (X) has a future value that exceeds another portfolio's (Y) future value, then a "monotonic" risk measure will be lower for portfolio X than for Y, indicating that portfolio Y is riskier. Translation invariance refers to the proportionate decrease in risk as a certain quantity of cash, or a risk-free asset, is added to a portfolio.

The limitations of VaR were tolerated, despite being acknowledged, until certain events such as the credit crisis of 2007-2008 highlighted the understatement of potential losses prior to the crash. As a result, these shortcomings could no longer be overlooked, leading to the introduction of ES in the "Fundamental Review of the Trading Book" overhaul by the BCBS (BCBS, 2013). The ability of ES to evaluate tail risk and its proven subadditivity are noted as reasons for its emergence as a preferred risk measurement tool (in certain scenarios) over VaR. For any given portfolio, ES measures the probability weighted losses beyond VaR (Taylor, 2019). Then, by definition, VaR remains a crucial step in its computation and retains its significance in estimating market risk and associated measures. When focusing on methods with asymmetrical risk profiles, such as writing option contracts, ES efficiently captures the lowest possibility of suffering larger losses than what is expected. VaR, on the other hand, erroneously lowers risk estimations since it overestimates the size of prospective losses for such methods (Jorion, 2007).

Despite its limitations, VaR is often preferred over sub-additive risk measures such as ES by both industry and regulators in the banking sector due to its practical benefits, which include smaller data requirements, ease of backtesting, and, in some cases, ease of calculation (Daníelsson et al., 2012). Orhan and Köksal (2012) contend that despite research highlighting the lack of sub-additivity and convexity in VaR, the measure is still the most effective way to quantify risk. More recently, the emergence of complex financial derivatives and associated volatilities has called for the development of an indicator capable of handling the highly unpredictable nature of these regularly changing products (Adamko et al., 2015).

There is no singular approach that banking institutions are encouraged to adopt for estimating VaR, primarily because research has not identified an optimal method that outperforms others on a consistent basis for conducting such estimates. The BCBS do not prescribe a preferred method and favour each of three mentioned approaches (variance-covariance matrices, historical simulations, or Monte Carlo simulations) equally. According to the Basel II Capital Accord, banking institutions may use any model, so long as each model implement-ed can capture all the material risks faced by the company (BCBS, 1996).

These approaches, along with others that have been developed more recently, differ in terms of their computational and modelling complexity. This has led to trade-offs between methods that may offer optimal performance but require more resources to implement. Among the most popular methods to estimate VaR are the Historical, Variance-Covariance (VCV), and Generalised AutoRegressive Conditional Heteroskedasticity (GARCH) approaches. Within the VCV framework, two common methods for estimation include the Equally Weighted (VCV EW) and Exponentially Weighted Moving Average (VCV EWMA) approaches. Monte Carlo simulation, a well-recognised method, was not included in this study as it does not replicate the actual distribution of risk factors for a portfolio or index. The use of Monte Carlo simulation primarily relates to portfolios characterised by a significant concentration of derivatives, wherein the absence of pre-existing historical data necessitates the generation and simulation of historical prices. For instance, when considering an Over the Counter (OTC) derivative contract executed between two counterparties, the absence of a recorded price history necessitates the simulation of plausible historical price scenarios.

Storti and Wang (2022) proposed a new semi-parametric approach for estimating and forecasting expected shortfall (ES) based on quantile time series regressions and a parsimoniously parameterised β weight function. Their approach was found to outperform other parametric and non-parametric models in forecasting studies, including during the 2008 Global Financial Crisis.

Effective measurement of the performance of the Kalman filter in estimating VaR must consider the existing methodologies, as well as their associated assumptions and logical flaws. Most like this study was that of Berardi et al. (2002), who used the Kalman filter to calculate VaR by estimating portfolio βs, treating the β parameter as if it were unobservable and followed a first order autoregressive process.

Das (2019) introduced advancements in Adaptive Kalman Filters (AKFs) to address parameter inconsistency issues by incorporating adaptive noise covariances for estimating asset β and VaR. The empirical performance of the proposed filters was compared with the standard least square family and Kalman Filters, based on VaR backtesting, ES analysis, and in-sample forecasting. Results showed that the Modified AKFs perform on par with the benchmark methods, even when considering the adaptive noise covariance assumptions, suggesting that the proposed techniques offer a viable and effective approach for estimating β and VaR in financial applications.

Saidane (2022) presents a computationally efficient Monte Carlo-based latent factor modelling approach for estimating portfolio VaR using a Kalman filter with maximum likelihood estimation. The methodology allows for the calculation of model parameters and inferences about unobservable factors, their volatilities, and the hidden state sequence of the Markov process. The methodology is applied to real-world data and the results indicate that this new specification provides a good fit, leading to improved accuracy in predicting VaR. The model also demonstrates a reduction in the number and average size of back-testing breaches during financial crises, highlighting the potential of the proposed approach to enhance VaR estimation and risk management in financial markets.

Abbara and Zevallos (2023) introduce a novel approach for estimating and predicting asymmetric stochastic volatility models using dynamic linear models with Markov switching formulated as state space models. The likelihood is computed using Kalman filter outputs, and the parameter estimates are obtained through maximum likelihood estimation. The accuracy of the estimation is evaluated through Monte Carlo experiments and the proposed method is applied to real-life time series data in a backtesting exercise. The authors found it provided a fast and reliable alternative for forecasting VaR.

2.1. Historical Simulation

Historical estimation represents the simplest method of chosen approaches, assuming prices of assets behave in a similar manner to what has been witnessed in the past (Sharma, 2012). Current weights are applied to a time-series of historical asset returns, focusing on reconstructing the history of a hypothetical portfolio based on its current position (Adamko et al., 2015). The simplicity of the Historical approach, its ability to easily incorporate stress scenarios, the logical time horizon measurement period (based on length of holding time), and the omission of standard deviation or correlation requirements (stemming from an empirical loss distribution rather than an imposed one) have established the method as a compelling choice in industry. However, several key limitations are evident-older returns which are potentially irrelevant to the context of the current market are weighted the same as recent returns, and there is an increased requirement for historical data coupled with an inability to isolate short term data in contrast with other methods. Underlying changes in implicit volatility can also take longer periods of time to be realised with this approach (Adamko et al., 2015).

To address the limitations of the historical approach, Žiković and Filer (2009) compared the effectiveness of VaR and ES models using a hybrid Historical simulation. This analysis spanned the period before and after the 2008 financial crisis, encompassing both developed and emerging markets. The hybrid model employed a combination of nonparametric bootstrapping and parametric GARCH volatility forecasting. Through backtesting, the hybrid approach was found to offer equivalent protection to extreme value (EV) models, but with significantly lower capital reserve requirements. The hybrid approach was found to yield the smallest error statistics for ES, particularly in developed markets.

2.2. VCV EW Estimation

The general VCV method operates under the assumption that the risk factors influencing the portfolio's value follow a multivariate normal distribution. As a result, the fluctuations in the value of a linear portfolio follow a normal distribution (de Raaji and Raunig, 1999). This implies that the VaR output is a multiple of the standard deviation, and is given by:

$$VaR = -\alpha\sqrt{w'\sum w}$$

Where α is a scaling factor representing a given confidence interval (usually 1.65 at 95%, 1.96 at 97.5%, and 2.33 at 99%), w and w' denote a vector of absolute portfolio weights and its transpose respectively, and $\sum$ is a variance-covariance matrix (JP Morgan, 1996).

Compared to the historical approach and the previously mentioned Monte Carlo simulation, the VCV method offers a distinct advantage in allowing for the prediction of volatilities in financial returns (JP Morgan, 1996). Moreover, the method is straightforward to implement and has demonstrated satisfactory precision and accuracy, requiring fewer data compared to the Historical approach.

To apply the VCV method, an approximation of the covariance matrix of the risk factors is required. In most cases, the variances (and covariances) are computed based on the daily historical time series of returns for the corresponding risk factors, employing equally weighted moving averages:

$$\sigma_{ijT}^2 = \sum_{t=T-n}^{T-1} \frac{r_{it}r_{jt}}{n}$$

Where σ_{ijT}^2 is the variance (or covariance) at time T, n is the number of observations, and $r_{it}\, r_{jt}$ are the corresponding risk factor returns.

2.3. VCV EWMA Estimation

VCV EWMA differs from VCV EW in that current values are weighted more heavily than past values. JP Morgan (1996) define the EWMA estimator in its recursive form by:

$$\sigma_{ij/t}^2 = \lambda\sigma_{ij/t-1}^2 + (1-\lambda)r_{it-1}r_{jt-1}$$

Where λ measures the declining weighting scheme of observations, σ_{ijT}^2 is the variance (or covariance) at time t–1, and $r_{it-1}\, r_{jt-1}$ are previous day's returns. This weighting tilt allows for a faster reaction to market crashes or significant changes in a given economy. The determination of a suitable λ is non-trivial, involving calibration using many data from the relevant market. We used 10 years of daily share return data selected from stocks on the Dow Jones Industrial Average (DJIA - the same source as our data used for the VaR and ES calculations) using the recipe given in JP Morgan (1996). We found $\lambda = 0.935$, well within the historical range.

For the stock prices of a well-known multinational firm, Galdi and Pereira (2007) investigated the effectiveness of VaR estimation techniques for VCV EWMA, GARCH, and stochastic volatility (SV) across a sampled 1 500-observation window. Relative to more "sophisticated" methods in GARCH and SV, VSV EWMA did *not* produce inferior violation test results. Additionally, the model required less computational effort to implement.

Although the VCV methods discussed above have straightforward implementation, nonlinear financial instruments such as derivatives containing non-normal distributions of profit or loss are problematic for VCV calculations (Best, 2000). If the underlying risk factors are not normally distributed, finding their associated distribution is challenging.

2.4. GARCH Estimation

Like VCV EWMA, the GARCH approach is non-linear, yet differs through its ability to account for asset volatility reverting to a long-term mean (Poon and Granger, 2003). The magnitude of standard deviation, a key component of VaR, is effectively tracked by GARCH in comparison to other models. The GARCH formulation, derived as a generalisation of the autoregressive heteroscedasticity model (ARCH) was proposed by Bollerslev (1986) and is as follows:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{u}\alpha_i\alpha_{t-i}^2 + \sum_{j=1}^{\upsilon}\beta_j\sigma_{t-j}^2$$

Where the sum of α_i and β_i determines how persistent "shocks," or unexpected deviations, to volatility will be. A significant shock to volatility in a previous period (i.e., yesterday) raises the likelihood of a shock to volatility today, which is a helpful representation for the clustering in volatility that is typically seen in series of returns (Tsay, 2010). A limitation of this model is that it does not distinguish between the effects of positive and negative shocks, which frequently have different effects (Restrepo, 2012).

Like VCV EWMA, GARCH models imply serial correlations in the returns of financial assets. More recent results are favoured over earlier ones and as a result, both models estimate volatility based on the most recent return data (Best, 2000). So and Yu (2006) applied seven variations of the GARCH model to four foreign exchange rates and 12 market indices to evaluate VaR at varying confidence intervals. The findings suggest that both fractionally integrated and stationary GARCH models are more effective than RiskMetrics in predicting VaR at a 1% level.

Chinhamu et al. (2022) proposed a robust modelling framework using long memory models with specific distributions for precious metal prices to improve market risk management and assessment. The findings suggest that certain ARFIMA-GARCH models with heavy-tailed distributions are suitable for accurately estimating VaR and forecasting future volatility for effective risk management and portfolio strategies in the highly volatile metals market.

Letmathe et al. (2022) introduced new semiparametric GARCH models with long memory and applied these to obtain forecasts for VaR and ES for market risk assets. The results suggest that these models are a meaningful alternative to conventional, parametric models according to regulatory tests and model performance evaluation.

Patton et al. (2019) looked to address the challenge of "elicitability" in ES estimation by employing a joint modelling approach that incorporates both VaR and ES. By applying this approach to daily returns on four international equity indices, the joint model was found to outperform GARCH models in terms of forecasting accuracy.

2.5. Kalman Filter Estimation

The Kalman filter, proposed by Kalman (1960) and with origins in autonomous navigation processes and trajectory tracking, has more recently been applied to financial markets. In mathematical finance, the issue of estimating unobserved latent variables from observable market data regularly occurs. Calibrated to solve similar issues in engineering and econometrics, the Kalman filter is employed in applications for data smoothing as well as the construction of time series models for variable forecasting (Date and Ponomareva, 2010).

Berardi et al. (2002) established that VaR estimation using a Kalman methodology was both feasible and suitable. The authors employed a first-order autoregressive process to estimate portfolio βs in their approach. The sequence of values for $\beta_{i,t}$ could be estimated and the final $\beta_{i,T}$ was then calculated to derive the VaR for the portfolio. A portfolio comprising ten stocks traded on the Nasdaq stock market was examined. Initially, an equal percentage of investment was assumed for each asset, followed by a random simulation of 5 000 portfolio compositions. In both scenarios, the portfolio composition remained unchanged over time. Backtesting analysis revealed that the Kalman filter-based approach exhibited sensitivity to changes in market volatility, yielding notable and significant results.

Previous studies, such as the work conducted by Bernales et al. (2014), have demonstrated the effectiveness of the Kalman filter in calculating market risk measures. The Kalman methodology was applied to a thinly traded fixed income portfolio to assess its ability to provide appropriate risk measures for a market where the portfolio is traded infrequently. The methodology employed a three-stage process. Firstly, the Kalman filter was used to extract a complete price dataset, even in situations where there were only a few price observations available, allowing for prices to be estimated on days with limited price information. In the second stage, market risk measures, specifically VaR, were estimated using the complete price dataset obtained from the first stage. Finally, a back-test was conducted to verify the reliability of the Kalman approach in estimating the price model. The empirical evidence presented suggests that the Kalman filter approach provided reliable measures of VaR for securities that are traded infrequently. The Kalman approach also outperformed the conventional method of simply replicating the last traded price in calculating the chosen risk measure (Bernales et al., 2014).

Date and Bustreo (2015) proposed the Kalman filter as a method to measure VaR for sovereign debt portfolios by simulating bond prices with a two-factor short rate model. The Kalman approach only required a simulation of a vector of two random variables for one-step ahead forecasts, resulting in computational "cheapness" in comparison to principal components analysis which utilises more than two principal components. The results indicated an arguably more transparent and accurate reflection of market conditions associated with highly liquid government securities.

Fundamental Sharpe ratios were estimated by Gatfaoui (2016) using a Kalman filter approach. The Sharpe ratio is a measure of risk-adjusted performance for a portfolio or individual security, defined as follows:

$$SR_a = \frac{E[R_a - R_b]}{\sigma_a}$$

Where R_a is the asset return, R_b is the riskless asset return, and σ_a is the asset's excess return standard deviation. Contrary to risk measures of *loss* in VaR and ES, this research focused on assessing the accuracy of risk-adjusted *performance* estimation using the Kalman filter. To account for the time variation, idiosyncratic risk, and market trend bias, the Sharpe ratios were adjusted into filtered Sharpe ratios (FSRs). The FSRs were designed to isolate the fundamental component of the Sharpe ratio on a time series basis. By applying the Kalman filter methodology, the time varying FSRs were captured, thereby excluding any previous biases inherent in the metric. Thereafter, a comparative analysis of various modelling techniques was performed, including GARCH and Monte Carlo simulations. Equally weighted portfolios were constructed, incorporating the highest performing equities identified by each measure. The FSR portfolio, when compared to the comparable portfolios, demonstrated reduced VaR forecasts and higher expectations of gains. This research showcases the capabilities of the Kalman filter in extracting fundamental Sharpe ratios, which are free from bias and serve as pure performance indicators, distinct from the traditional Sharpe ratios.

Thomson and van Vuuren (2018) decomposed the time series of hedge fund returns into market timing and stock selection factors using the Kalman filter. Representing the first application of Kalman in this manner, the model was used to determine whether statistically significant abnormal profits are truly generated by hedge fund managers, in accordance with popular belief. Through an extension of the capital asset pricing model (CAPM) equation, stock selection, market timing, and market exposure components may be separated from hedge fund results. The authors conclude that one could use the Kalman filter, which employs Bayesian variance reduction, to get the parameters required for this enhanced CAPM. The paper found that top-performing hedge funds obtained the majority of their α from consistent stock selection and somewhat from market timing. These funds also showed less fluctuation in return. The worst-performing funds had variable market timing α and greater volatility, implying that attempts to time the market frequently cause volatility and reduce long-term returns.

Das (2019) employed an adaptive Kalman filter approach to effectively track and estimate the market risk β and VaR in the Indian market. This approach did not rely on assuming the noise covariance (i.e., uncertainties). The adaptive Kalman filter demonstrated similar performance to an ordinary filter, reinforcing previous observations that sector β estimates are dynamic and not constant in nature. Das (2019) presents recent findings that highlight the efficacy of utilising the mathematical principles of Kalman in accurately estimating VaR.

Van Rooyen and van Vuuren (2022) explored asset allocations using the Kalman filter, estimating α and β parameters as they appear in the CAPM to forecast asset returns. Two approaches in Tactical Asset Allocation (TAA) and Strategic Asset Allocation (SAA) were examined in the paper. To forecast asset returns, TAA uses quantitative methods, notably the CAPM framework and estimations using the Kalman filter. By dynamically altering asset class weights based on the anticipated returns, this strategy seeks to enhance portfolio performance and risk characteristics. The results indicate that, when compared to a "static" SAA allocation, the TAA strategy, which makes use of the Kalman filter and dynamic asset allocation, can improve portfolio performance and risk characteristics.

Claver et al. (2023) used the Kalman filter to develop a dynamic system for predicting price movements of a single equity. By simulating the equity's movement using the filter, price levels were forecasted with greater accuracy relative to more traditional approaches.

3. METHODOLOGY AND DATA

3.1. Kalman Filter

The Kalman filter is a Bayesian updating method designed to optimise the accuracy of estimating unknown parameter values (Koch, 2006). This filter deals with the broader issue of estimating the state $\left[x \in \Re^n\right]$ of a discrete, time-controlled process that follows a linear stochastic difference equation as follows:

$$x_t = Fx_{t-1} + Bu_{t-1} + w_{t-1} \quad (1)$$

With a measurement $\left[x \in \Re^n\right]$:

$$z_t = Hx_t + v_t \quad (2)$$

Where F denotes the state transition matrix responsible for transitioning between states, B represents the control matrix that maps control variables to state variables, and H represents the measurement matrix responsible for mapping measurements onto the state.

The random variables w and v denote process white noise and measurement white noise, respectively. It is assumed that these variables are independent of each other, meaning there is no correlation between them. Both w and v are assumed to follow normal probability distributions: $w(\cdot)\sim N(0,Q)$ and $v(\cdot)\sim N(0,R)$.

In practical applications, the covariance matrices Q and R, which represent the process noise and measurement noise respectively, may vary at each time step. However, in this context, they are assumed to remain constant, as stated by Koch (2006), estimated using maximum likelihood methods.

The state transition matrix F, with dimensions 2 × 1 in this case, connects the state at the previous time step t–1 to the current state at step t, assuming the absence of any driving function or process noise. On the other hand, the control matrix B, with dimensions 2 × 2, establishes the relationship between the optional control input $u \in R^1$ and the state x. Additionally, the 2 × 1 matrix H in the measurement describes the relationship between the state and the measurement z_k. Although in practice, F and H may vary with each time step, in this scenario, both matrices are assumed to remain constant.

The intended procedure for the mechanical process is as follows:

Predict		
Project state 1 time step ahead	$\hat{x}_{t\mid t-1} = F_t\hat{x}_{t-1\mid t-1} + B_t u_t$	(3)
Project error covariance 1 step ahead	$P_{t\mid t-1} = F_t P_{t-1\mid t-1} F_t^T + Q_t$	(4)
Update		
Compute Kalman gain	$K_t = P_{t\mid t-1} H_t^T \left(H_t P_{t\mid t-1} H_t^T + R_t\right)^{-1}$	(5)
Update estimate with measurement yt	$\hat{x}_{t\mid t} = \hat{x}_{t\mid t-1} + K_t\left(y_t - H_t\hat{x}_{t\mid t-1}\right)$	(6)
Update error covariance	$P_{t\mid t} = \left(I - K_t H_t\right) P_{t\mid t-1}$	(7)

Where $\hat{x}$ represents the estimated state, F denotes the state transition matrix responsible for transitioning between states, u represents the control variables, B represents the control matrix that maps control variables to state variables, P represents the state variance matrix, Q represents the process variance matrix that captures errors caused by the process, y represents the measurement variables, H represents the measurement matrix responsible for mapping measurements onto the state, K represents the Kalman gain, and R represents the measurement variance matrix that accounts for errors originating from measurements.

Subscripts represent:
t|t: Current time
t–1|t–1: Previous time, and
t|t–1: Intermediate steps.

The observation equation is the VaR, which can be expressed as follows:

$$VaR_{CI}^{1d}(t) = \mu(t) + \sigma(t) \cdot N^{-1}(CI) + \epsilon(t)$$
$$\epsilon(t) \sim N(0, \sigma_\epsilon^2) \quad (8)$$

Where μ represents the average of daily returns over the previous period, σ denotes the daily standard deviation of the portfolio or security return, CI is the confidence interval, and ϵ represents a noise term. The noise term ϵ is assumed to be independently and identically distributed (i.i.d.) with a normal distribution $\sim N(0,\sigma_\epsilon^2)$, where $0,\sigma_\epsilon^2$ represents the variance of ϵ.

The specific form of the transition equation depends on the stochastic process assumed for the time-varying αs and βs. It can be modelled using either an autoregressive, mean-reverting (AR[1]) model or a random walk process. Research has shown that the random walk model provides a more robust characterisation of time-varying βs (Denrell, 2004). On the other hand, AR(1) forms of the transition equation may encounter convergence issues, which can indicate misspecification of the transition equation, particularly for certain return series (Faff et al., 2000).

The random walk model (RWM) assumes that both α and β follow a random walk process. In other words, the current market exposure is considered a normally distributed random variable, with its mean being the exposure of the previous period. The uncorrelated system noises, including the evolution of α and β, are also assumed to be normally distributed.

The state variables $x(t) \in \mathfrak{R}^2$ are the time-varying coefficients:

$$x(t) = \begin{bmatrix} \mu(t) \\ \sigma(t) \end{bmatrix}$$

At each time t. Both are assumed to follow the random walk model. The state equation is:

$$\begin{bmatrix} \mu(t+1) \\ \sigma(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mu(t) \\ \sigma(t) \end{bmatrix} + \begin{bmatrix} \gamma \\ \delta \end{bmatrix} \quad (9)$$

Where

$$\begin{bmatrix} \gamma \\ \delta \end{bmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \sigma_\gamma^2 & 0 \\ 0 & \sigma_\delta^2 \end{bmatrix} \right)$$

And the measurement equation is:

$$VaR_{CI}^{1d}(t) = \begin{bmatrix} 1 & CI(t) \end{bmatrix} \begin{bmatrix} \mu(t) \\ \sigma(t) \end{bmatrix} + \epsilon(t) \quad (10)$$

3.2. ES

The ES at a selected quantile, q, denoted as ES_q, is computed as the probability-weighted average of values in the tail below q, such that:

$$ES_q = E\left(L | L < VaR_q\right)$$

For a normal distribution,

$$ES_q = \frac{f\left(VaR_q\right)}{q}$$

where

$$f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

In other words, the probability density of the normal distribution is used to calculate ESq, where σ_t is the volatility. The function $f(x)$ refers to the probability density function of the normal distribution $N(0,\sigma^2)$, and it is assumed that $\mu = 0$.

To calculate ESq for any volatility, σ, and at any significance level, q, the function below must be integrated:

$$ES_q \Big| = \int_{-\infty}^{q} x \cdot f(x) dx$$
$$\Big| = \int_{-\infty}^{q} \frac{x}{\sqrt{2\pi} \cdot \sigma} \cdot \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$$
$$\Big| = -\frac{\sigma}{\sqrt{2\pi}} \cdot \exp\left(-\frac{q^2}{2\sigma^2}\right)$$

The observation that the distribution of asset price returns has fatter tails compared to a normal distribution has led to the introduction of the student's-t distribution. This distribution is employed to more accurately model the excessive kurtosis observed in asset returns (Fama, 1965; Bekaert et al., 1998). When calculating the ES for a portfolio using the student's-t distribution, the integration process follows a similar procedure as with the normal distribution.

$$ES_q \Big| = \int_{-\infty}^{q} t \cdot f(t) dt$$

In this case, $f(t)$ is the probability density function of the t -distribution, which is (for $\mu = 0$ and standard deviation, σ):

$$f(t) = \frac{\Gamma(v+1)}{\sqrt{v\pi} \cdot \Gamma\left(\frac{v}{2}\right) \cdot \sigma} \left(1 + \frac{t^2}{\sigma^2 v}\right)^{-\left(\frac{v+1}{2}\right)}$$

Where v counts the degrees of freedom, calculated using $k = \frac{6}{v-4} + 3$ and where k is the kurtosis of the data (Rozga and Arnerić, 2009)

For v even: $$\frac{\Gamma(v+1)}{\sqrt{v\pi}\cdot\Gamma\left(\frac{v}{2}\right)}=\frac{(v-1)\cdot(v-3)\cdots5\cdot3}{2\sqrt{v}(v-2)\cdot(v-4)\cdots4\cdot2}$$

And for v odd: $$\frac{\Gamma(v+1)}{\sqrt{v\pi}\cdot\Gamma\left(\frac{v}{2}\right)}=\frac{(v-1)\cdot(v-3)\cdots4\cdot2}{\pi\sqrt{v}(v-2)\cdot(v-4)\cdots5\cdot3}$$

To calculate ESq for any volatility, σ, any number of degrees of freedom, v, and any significance level, q, the integral below must be determined:

$$\begin{aligned} ES_q &= \int_{-\infty}^{q} t\cdot\frac{\Gamma(v+1)}{\sigma\sqrt{v\pi}\cdot\Gamma\left(\frac{v}{2}\right)}\cdot\left(1+\frac{t^2}{\sigma^2 v}\right)^{-\left(\frac{v+1}{2}\right)}dt \\ &= \frac{\Gamma(v+1)}{\Gamma\left(\frac{v}{2}\right)}\cdot\left(\frac{\sigma}{1-v}\right)\cdot\sqrt{\frac{v}{\pi}}\cdot\left(1+\frac{q^2}{\sigma^2 v}\right)^{\frac{1-v}{2}} \end{aligned}$$

3.3. Data

The data-sourced from Bloomberg-comprised daily returns of 30 liquid ($400 bn ≤ market capitalisation ≤ $200 bn) companies selected from the DJIA, covering the period from January 2012 to April 2023. These individual stocks were assembled into a single portfolio (on whose returns the results are based). Data from the same period were sourced from three other indices: two from developed economies (UK FTSE 100 and German DAX 40) and one from a developing economy (South African Johannesburg All Share top 40). Because the results were like those produced from the DJIA, we have excluded them from this article for the sake of brevity.

This time frame was selected to reflect roughly two US business cycles which are ≈5.5 years (National Bureau of Economic Research, 2023) and to encompass the pre-Covid-19 period (financial stability), the Covid-19 period (considerable market turbulence) and then a post-Covid period of growth and recovery. Other significant global events having recent influence on financial markets have been included, such as the Russian invasion of Ukraine, which began in February 2022 and continues at the time of writing (August 2023) with no resolution pending. The first 6 years (2012-2018) were used to train and calibrate the Kalman filter model to estimate parameters required for the remaining (out of sample) period.

Daily returns are used to create a daily estimation of VaR, as recommended by the Bank for International Settlements (BIS), for back testing analyses focused on assessing the differences in a VaR model output and the selected portfolio value on an ex-post basis (BIS, 2019). Using a 97.5% confidence interval, the analysis measures whether daily losses beyond VaR are experienced 2.5% of the time - in accordance with the Basel II Capital Accord requirements. Rather than extracting data on individual equities, this approach used data on the prices of a *single* popular index (the DJIA). An index is frequently recalculated, and its composition varies constantly over time, which eliminates inactive equities and reduces the likelihood of survivorship bias in equity selection. To conform with its goals as an index, the DJIA also changes and redistributes weights accordingly.

This procedure was applied, and estimates were computed for the Historical, VCV EW, VCV EWMA, GARCH, and Kalman filter methods.

4. RESULTS

Figures 1 and 2 illustrate the rebased index prices and daily returns for the DJIA over the period from Jan-17 to Apr-23 (prices rebased in Jan-17 to 100) respectively. Figure 1 displays elevated volatility between Jan-20 and Jul-20, which can largely be attributed to the influence of the Covid-19 pandemic on global financial markets. This period may be classified as an "extreme" event, wherein risk measures such as VaR and ES hold significant potential in mitigating substantial losses for given institutions and banks. The daily returns of the DJIA have fluctuated within a range of 10% (comprising a 5% positive return and a 5% negative return) both preceding and following the aforementioned period of pronounced volatility. The focus will thus be on the effectiveness of each measure compared with the Kalman filter.

The different methods selected to estimate the VaR and ES for the DJIA over the period were compared against the Kalman estimation approach. The results of these estimations are presented in Figures 3-6, respectively. Each figure plots the different VaR and ES estimation approaches at the 97.5th percentile. Each point represents a *negative* daily return on the DJIA over the period, as VaR and ES are only concerned with downside risk. A return that falls below a line indicates that the estimation technique was unsuccessful in forecasting market risk and this point reflects an "exceedance."

Throughout the period, the trajectory of Historical VaR aligns with that of Kalman VaR in Figure 3. The Kalman approach takes an additional step by using variance reduction techniques to mitigate noise and produce instantaneous measures (Thomson and van Vuuren, 2018).

All approaches respond rapidly to the market downturn; however, they fall short in capturing the most severe losses that transpired, especially during the specific period from Mar-20 to Aug-20. The Historical approach performs equally as well as the Kalman approach during this isolated period, while exhibiting slightly more sensitivity.

The VCV EW approach exhibits a comparatively lower level of sensitivity to the "extreme" event. Like the Historical VaR, the VCV EW VaR fails to capture as many negative returns as the Kalman approach, both in the post-Apr-21 period and the pre-Jan-20 period.

Contrary to the Historical and VCV EW approaches, the VCV EWMA and GARCH techniques exhibit a pronounced increase in VaR estimation sensitivity in response to fluctuations in returns.

Figure 1: Times series of the DJIA, rebased to 100 in Jan-17

Figure 2: Daily return series of the DJIA

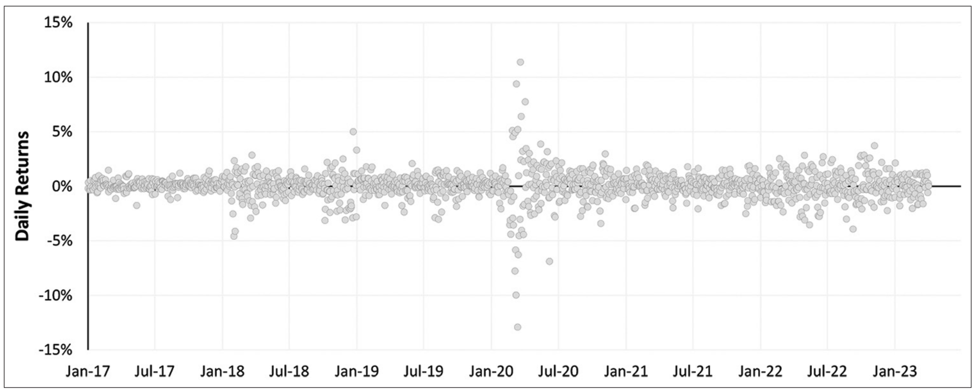


Figure 3: Comparison of VaR estimation approaches using Historical and VCV EW tech-niques with the Kalman filter method

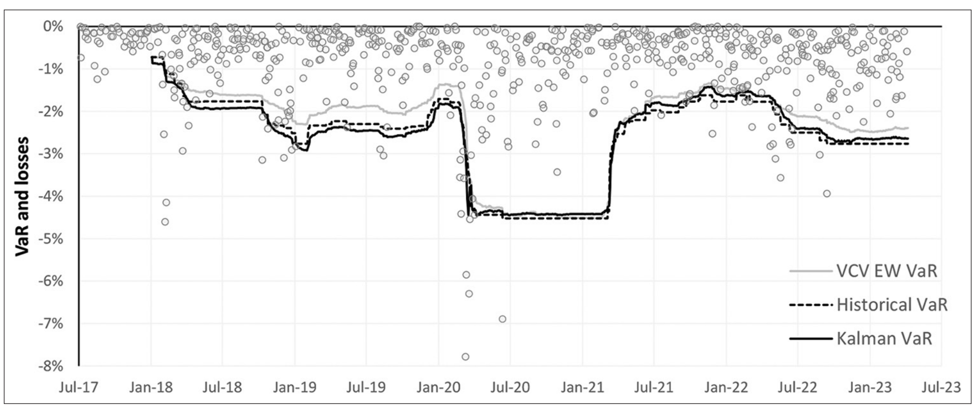


Figure 4: Comparison of VaR estimation approaches using VCV EWMA and GARCH tech-niques with the Kalman filter method

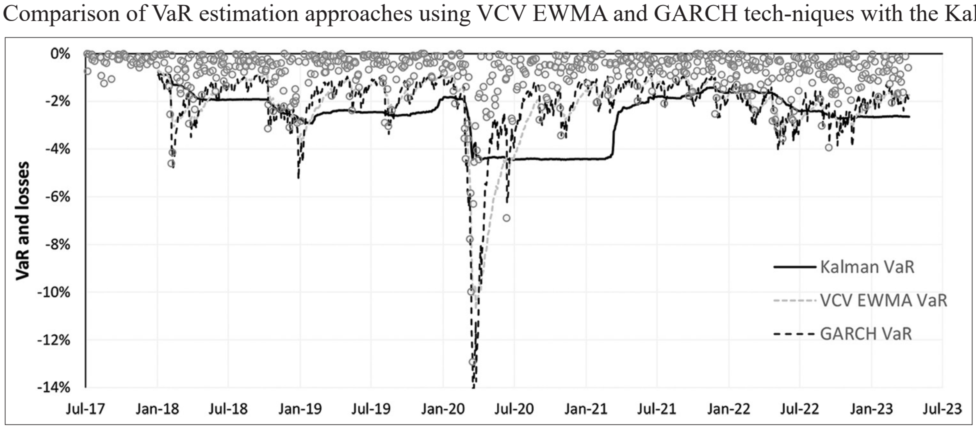


Figure 5: Comparison of ES estimation approaches

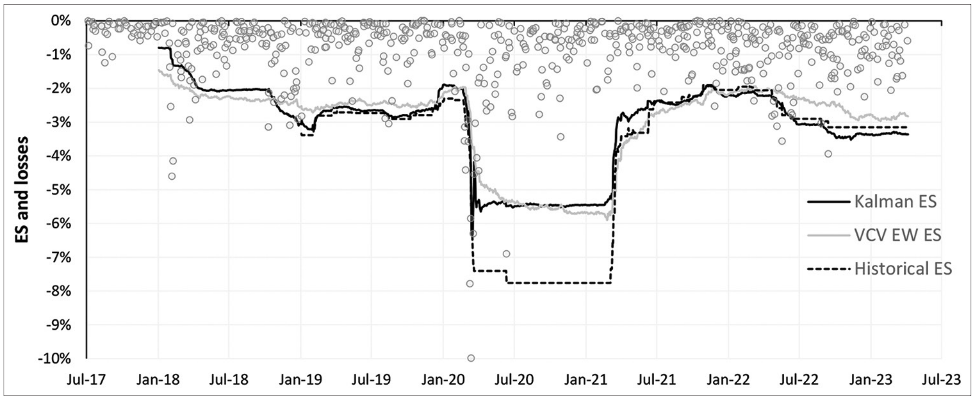
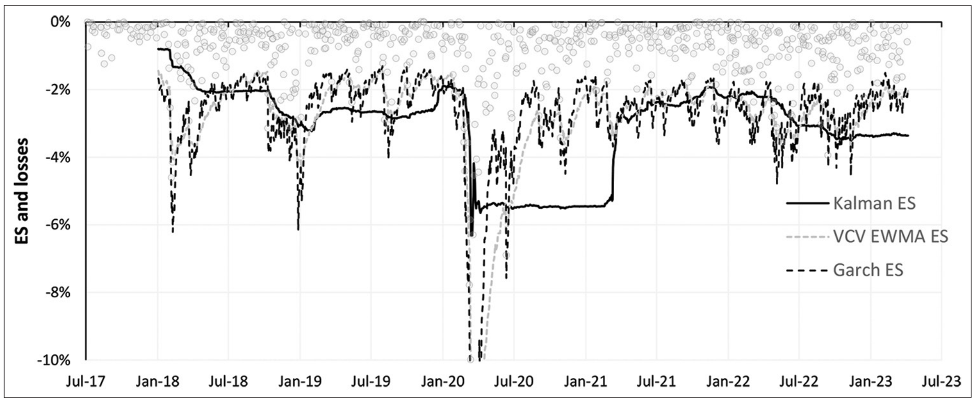


Figure 6: Comparison of ES estimation approaches (truncated y axis)



This effect is prominent during the period of high impact caused by the Covid-19 pandemic: both methods show a more pronounced reaction compared to the Kalman approach. The simultaneous perturbations, evident with GARCH, indicate a heightened sensitivity to extreme events compared to the Kalman estimation. While this sensitivity can be advantageous in highly volatile circumstances, it noticeably leads to inadequate forecasts and captures of market risk, as evidenced by the respective timeframes before and after the most volatile Covid-19 market period where the methods failed to capture exceedances.

Comparing Figure 3, which illustrates the Kalman filter alongside Historical VaR and VCV EW VaR, the positioning of Historical ES has moved downwards in relation to Kalman ES in Figure 5. Historical ES is computed by averaging VaR values, meaning that the average only changes when the VaR value changes. Consequently, a new set of returns is averaged until the VaR changes again. The VCV EW approach demonstrates a relatively lower sensitivity to extreme events.

Figure 6 presents a comparison of Kalman ES, GARCH ES, and VCV EWMA ES, which exhibits a similar pattern to their respective VaR counterparts in Figure 4, albeit on a larger scale. Once again, both the VCV EWMA and GARCH techniques demonstrate a more noticeable increase in sensitivity when estimating ES in response to returns fluctuations. However, in this case, all ES approaches have captured a greater number of returns before and after the volatile Covid-19 period.

Figure 7 shows VaR measured during high volatility and low volatility periods, respectively. The sensitivities of the VCV EWMA and GARCH methods are evident.

Figure 8 provides a summary of all VaR estimation approaches used over the entire period in relation to the Kalman estimation method while Figure 9 provides a summary of all ES estimation approaches used over the entire period in relation to the Kalman estimation method.

Figure 10 presents a cumulative count of VaR exceptions for each method. To determine these exceptions, the VaR forecast from the previous day is compared with the current day's return using the time series of VaR outputs and DJIA daily returns. If yesterday's forecast, which represents the amount set aside based on the VaR estimate, is smaller than the current day's return, it is recorded as an exception. The values obtained for and used in Figure 10 were tested using approaches detailed in Zhang and Nadarajah (2018) such as Kupiec's POF (1995) test, the binomial distribution test, the generalised Markov test (Pajhede, 2015) and the multivariate autocorrelations test (Hurlin and Tokpavi, 2006). All were found to be significant at the 99% confidence level.

Where the graph of a given model crosses above the 2.5% horizontal line over time, the number of exceptions has surpassed the anticipated level at that specific moment. This is unexpected considering that the computed VaR is designed to capture losses with a 97.5% confidence level. Ideally, a model should accurately achieve this, but during periods of significant volatility, models

Figure 7: VaR comparison during high volatility (left panel) and low volatility (right panel) period

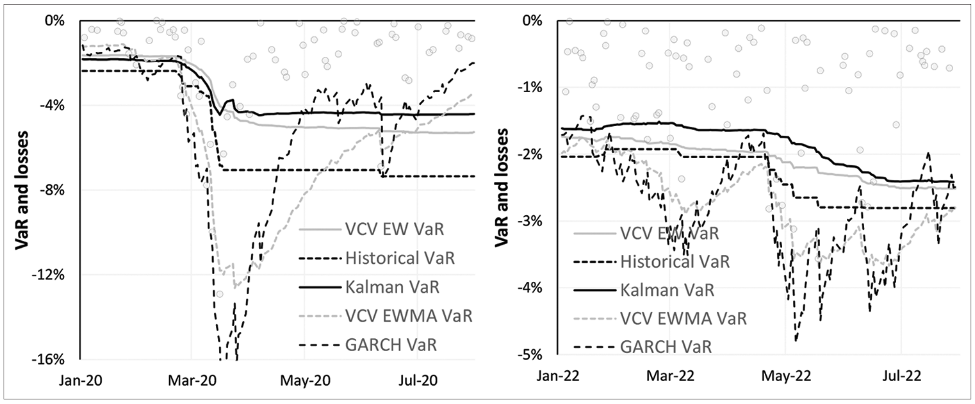


Figure 8: Efficiency comparison of popular VaR estimation techniques (truncated y axis)

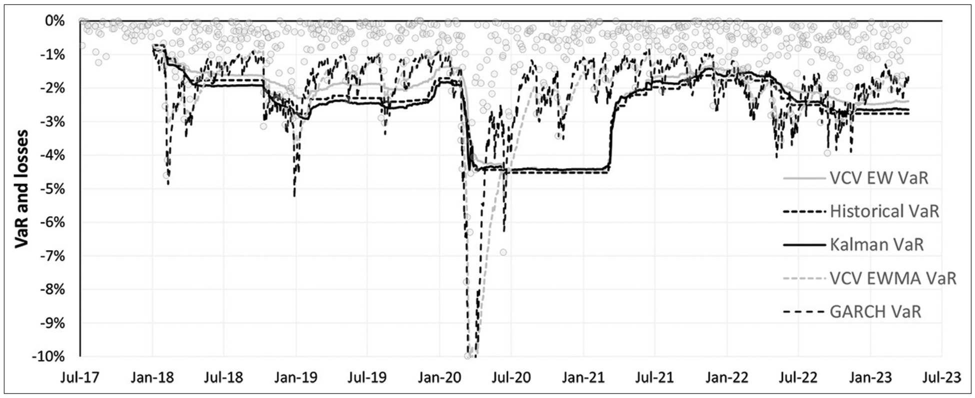


Figure 9: Efficiency comparison of ES estimation techniques (truncated y axis)

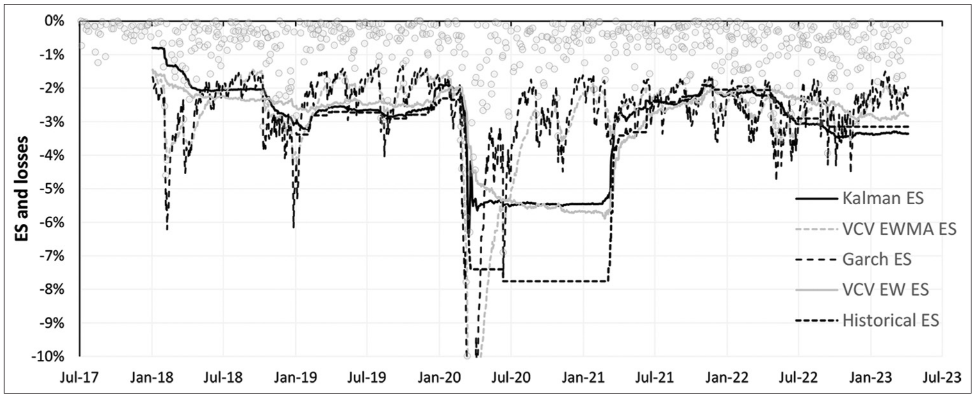


Figure 10: Cumulative count of VaR exceptions in relation to the Kalman method

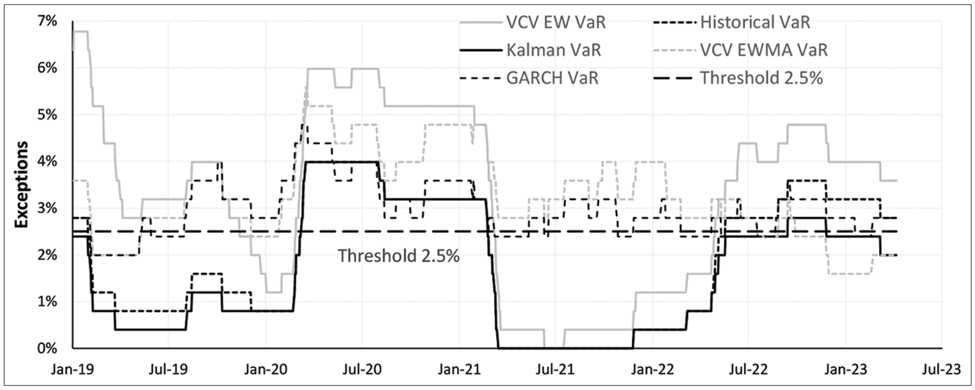


Figure 11: Comparison of VaR estimation inaccuracies relative to the Kalman method

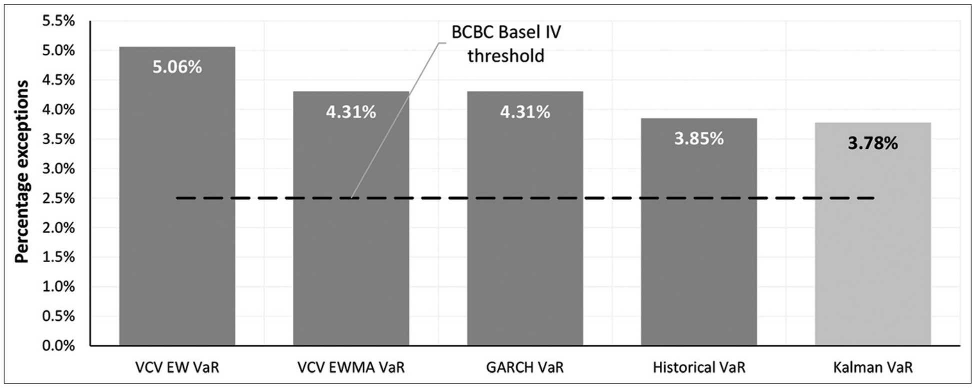
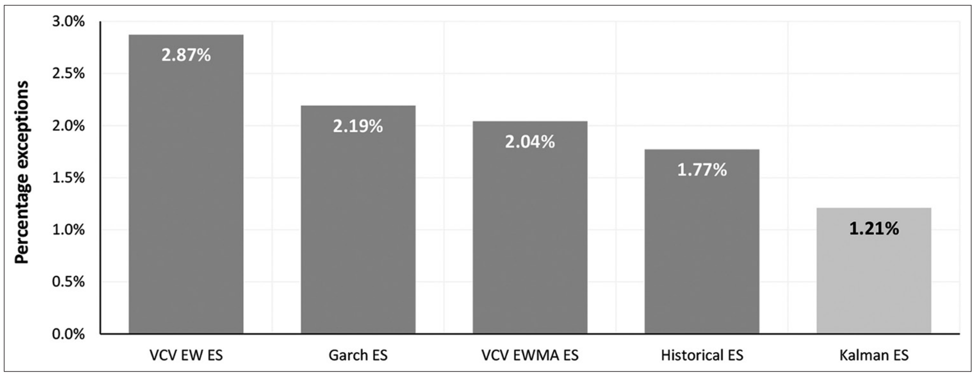


Figure 12: Comparison of ES estimation inaccuracies relative to the Kalman method



do not adjust rapidly enough, leading to significant increases in the number of exceptions. Models reflecting greater sensitivity (or conservativeness) are therefore those with fewer instances of exceeding the 2.5% threshold; the higher the number of exceptions, the less conservative the model's forecasting performance. Kalman and Historical VaR methods exhibit superior performance, with Kalman outperforming the Historical method during the Feb-19 to Aug-19 period. The VCV EW VaR, although briefly dipping below the 2.5% threshold between Mar-21 to Nov-21, demonstrates significant volatility before and after this period of relative stability.

Figure 11 provides an overview of the percentage of unsuccessful VaR forecasts by each method (i.e., the number of times the VaR forecast was insufficient to protect against the return (losses) the following day. Recall that for a perfect model, 2.5% accuracy is assumed. The Kalman filter estimation outperformed all approaches, followed closely by Historical VaR.

Figure 12 provides an overview of the percentage of unsuccessful ES forecasts by each method.

The ES forecast replaces the old VaR forecast level and as such represents the best guess for capital required over the next day (as per the BCBS backtesting requirements). If that amount is exceeded, the ES was insufficient, representing a "failure." The lowest percentage of failures thus reflects the most accurate approach. Among all the approaches considered, the Kalman filter method exhibited the fewest inaccuracies, i.e., the Kalman filter ES forecast was only insufficient 1.21% of the time. This finding is promising and highlights the robust performance of the Kalman approach in estimating both VaR and ES measures. The Kalman approach stands out for its ability to achieve such accuracy while imposing minimal limiting assumptions as discussed in Section 3 in relation to the other approaches, further enhancing its appeal and robustness in risk estimation.

5. CONCLUSIONS AND SUGGESTIONS FOR FUTURE STUDY

5.1. Practical Conclusions

The Kalman filter approach shows superior performance compared to commonly used approaches when assessing VaR and ES risk measures. This implies that financial institutions and risk managers could consider adopting the Kalman VaR approach to improve their risk assessment methods (Koch, 2006).

The Kalman estimation method is responsive to changes in market volatility, indicating that it can adapt well to dynamic market conditions. This responsiveness is a desirable feature for risk models as it allows them to capture the changing nature of financial markets accurately.

The Kalman filter is deemed reliable and robust in the context of volatility modelling. Its capacity to enhance the precision of estimating unknown parameter values makes it a valuable tool in financial risk management.

5.2. Policy Conclusions

Regulators and policymakers in the financial sector should be aware of the advancements in risk assessment methodologies, such as the Kalman VaR approach. Encouraging the adoption of more accurate and sophisticated risk models can contribute to better risk management practices within financial institutions.

When formulating risk management guidelines and regulations, policymakers should consider incorporating approaches that account for changes in market volatility. This can help create more adaptive risk frameworks capable of dealing with various market conditions.

5.3. Theoretical Conclusions

The study supports the existing literature that highlights the superiority of Expected Shortfall (ES) over VaR in accurately forecasting market losses. This adds to the theoretical understanding of risk measures, emphasizing the importance of ES as a more robust tool for capturing tail risk (Albanese, 1997; Artzner et al., 1999; Acerbi and Tasche, 2002; Jorion, 2007; Daníelsson et al., 2012).

The research demonstrates the applicability and efficacy of the Kalman filter in the context of financial risk measurement. This contributes to the body of knowledge regarding volatility modelling techniques and may inspire further research on advanced statistical methods for risk assessment.

5.4. Suggestions for Future Work

To better understand the effectiveness of the Kalman filter approach in different market environments, further research could extend this methodology to various financial markets. Each market is exposed to unique risks stemming from factors such as currency fluctuations, political landscapes, and market compositions.

Other volatility estimation approaches could be considered, such as GARCH (specifically GJR-GARCH). Also, alternate distributions could be considered, such as returns which follow t-distributions or skewed t-distributions.

Application of the Kalman filter in risk management can also be extended to other popular measures, an example of which being extreme value theory. In addition to traditional volatility estimation approaches, the evaluation of the Kalman filter approach can be extended to more advanced machine learning techniques. Models comprised of deep learning, support vector machines, random forests, and many others have gained prominence in recent years. Assessing the performance of the Kalman filter against these modern methodologies would provide a comprehensive comparison and highlight the relative strengths and weaknesses of each approach in volatility estimation and risk management.

REFERENCES

Abbara, O., Zevallos, M. (2023), Maximum likelihood inference for asymmetric stochastic volatility models. Econometrics, 11(1), 1-10.

Acerbi, C., Tasche, D. (2002), Expected shortfall: A natural coherent alternative to value at risk. Econ Notes, 31(2), 379-388.

Adamko, P., Spuchľáková, E., Valášková, K. (2015), The history and ideas behind VaR. Procedia Economics and Finance, 24(1), 18-24.

Albanese, C. (1997), Credit Exposure, Diversification Risk and Coherent VaR. Working Paper, Department of Mathematics, University of Toronto.

Arnold, T., Bertus, M., Godbey, J. (2008), A simplified approach to understanding the Kalman filter technique. The Engineering Economist, 53(2), 140-155.

Artzner, P., Delbaen, F., Eber, J.M., Heath, D. (1999), Coherent measures of risk. Mathematical Finance, 9(3), 203-228.

BCBS. (1996), Amendment to the capital accord to incorporate market risks. Available from: bis.org/publ/bcbs24.pdf

BCBS. (2013), Fundamental Review of the Trading Book: A Revised Market Risk framework. Available from: https://www.bis.org/publ/bcbs265.pdf

Bekaert, G., Erb, C.B., Harvey, C.R., Viskanta, T.E. (1998), Distributional characteristics of emerging market returns and asset allocation. J Portfolio Manage, 24(2), 102-116.

Berardi, A., Corradin, S., Sommacampagna, C. (2002), Estimating Value at Risk with the Kalman Filter. Available from: https://www.quantlabs.net/academy/download/free_quant_instituitional_books_/Estimating_Value_at_Risk_with_Kalman_Filter_Berardi_Corradin_Sommacampagna.pdf

Bernales, A., Beuermann, D.W., Cortazar, G. (2014), Thinly traded securities and risk management. Estudios Economía, 41(1), 5-48.

Best, P. (2000), Implementing Value at Risk. John Wiley and Sons: Hoboken, New Jersey. Available from: https://www.wiley.com/en-us/implementing+value+at+risk-p-9780470865965

BIS. (2019), MAR30-Internal Models Approach. Available from: https://www.bis.org/basel_framework/chapter/MAR/30.htm?inforce=20191215&published=20191215

Bollerslev, T. (1986), Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics, 31(3), 307-327.

Chen, J.M. (2014), Measuring market risk under the Basel accords: Var, Stressed Var, and Expected Shortfall estimation. The IEB International Journal of Finance, 8(1), 184-201.

Chinhamu, K., Chifurira, R., Ranganai, E. (2022), Value-at-Risk estimation of precious metal returns using long memory GARCH models with heavy-tailed distribution. Journal of Statistics Applications and Probability, 11(1), 89-107.

Claver, J.H., Dave, M., Che, S.F. (2023), Kalman filtering for stocks price prediction and control. Journal of Computer Science, 19(6), 739-748.

Daníelsson, J., Jorgensen, B.N., Samorodnitsky, G., Sarma, M., De Vries, C.G. (2012), Fat tails, VaR and subadditivity. Journal of Econometrics, 172(2), 283-291.

Das, A. (2019), Performance evaluation of modified adaptive Kalman filters, least means square, and recursive least square methods for market risk beta and VaR estimation. Quantitative Finance and Economics, 3(1), 124-144.

Date, P, Bustreo, R. (2015), Value-at-Risk for fixed-income portfolios: A Kalman filtering approach. IMA Journal of Management Mathematics, 27(4), 557-573.

Date, P, Ponomareva, K. (2010), Linear and non-linear filtering in mathematical finance: A review. IMA Journal of Management Mathematics, 22(3), 195-211.

De Raaji, G., Raunig, B. (1999), A comparison of value at risk approaches and their implications for regulators. Available from: https://semanticscholar.org/paper/a-comparison-of-Value-at-Risk-Approaches-and-Their-Raaji-Raunig/f3ed258d8f914cc04986789d728a8c68de651c4a

Denrell, J. (2004), Random walks and sustained competitive advantage. Management Science, 50(7), 922-934.

Faff, R.W., Hillier, D, Hillier, J. (2000), Time varying beta risk: An analysis of alternative modelling techniques. Journal of Business Finance and Accounting, 27(5-6), 523-554.

Fama, E.F. (1965), The behavior of stock-market prices. The Journal of Business, 38(1), 34-105.

Galdi, F.C, Pereira, L.M. (2007), Value at risk using volatility forecasting models: EWMA, GARCH and stochastic volatility. Brazilian Business Review, 4(1), 74-117.

Gatfaoui, H. (2016), Estimating Fundamental Sharpe Ratios: A Kalman Filter Approach. Available from: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2838935

Hurlin, C., Tokpavi, S. (2006), Backtesting value-at-risk accuracy: A simple new test. Journal of Risk, 9(2), 19-37.

Jorion, P. (2007), Value at Risk: The new benchmark for managing financial risk. 3rd ed. New York: McGraw-Hill. p114-118.

JP Morgan. (1996), Riskmetrics technical document. JP Morgan/Reuters. Available from: https://www.msci.com/documents/10199/5915b101-4206-4ba0-aee2-3449d5c7e95a

Kalman, R.E. (1960), A new approach to linear filtering and prediction problems. ASME Journal of Basic Engineering, 82(1), 35-45.

Koch, W. (2006), Advanced target tracking techniques. Advanced Radar Signal and Data Processing, 2(1), 2-34.

Krause, T.A., Tse, Y. (2016), Risk management and firm value: Recent theory and evidence. International Journal of Accounting and Information Management, 24(1), 56-81.

Kupiec, P. (1995), Techniques for verifying the accuracy of risk measurement models. Journal of Derivatives, 2(1), 73-84.

Letmathe, S., Feng, Y., Uhde, A. (2022), Semiparametric GARCH models with long memory applied to value-at-risk and expected shortfall. Journal of Risk, 25(2), 1-32.

Manganelli, S., Engle, R.F. (2001), Value at Risk Model in Finance. Vol. 75. Germany: European Central Bank Working Paper Series. p1-40.

Markowitz, H. 1952. Portfolio selection. The Journal of Finance, 7(1): 77-91.

Marshall, C., Siegel, M. (1997), Value-at-Risk: Implementing a risk measurement standard. Available from: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=1212

Miletic, M., Miletic, S. (2015), Performance of value at risk models in the midst of the global financial crisis in selected CEE emerging capital markets. Economic Research-Ekonomska Istraživanja, 28(1), 132-166.

National Bureau of Economic Research. (2023), US Business Cycle Expansions and Contractions. Available from: https://www.nber.org/research/data/us-business-cycle-expansions-and-contractions

Orhan, M., Köksal, B. (2012), A comparison of GARCH models for VaR estimation. Expert Systems with Applications, 39(3), 3582-3592.

Pajhede, T. (2015), Backtesting Value-at-Risk: A Generalized Markov Framework. Working Paper. UK: Lancaster University.

Patton, A., Ziegel, J.F., Chen, R. (2019), Dynamic semiparametric models for expected shortfall (and Value-at-Risk). Journal of Econometrics, 211(2), 388-413.

Poon, S.H., Granger, C.W.J. (2003), Forecasting volatility in financial markets: A review. Journal of Economic Literature, 41(1), 478-539.

Restrepo, M. (2012), Estimating portfolio value at risk with GARCH and MGARCH models. Perfil de Coyuntura Económica, 19, 77-92.

Roy, A. D. Safety first and the holding of assets. Econometrica, 20(3): 431-449.

Rozga, A., Arnerić, J. (2009), Dependence between volatility, persistence, kurtosis and degrees of freedom. Investigacion Operacional, 30(1), 32-39.

Saidane, M. (2022), Switching latent factor value-at-risk models for conditionally heteroskedastic portfolios: A comparative approach. Communications in Statistics: Case Studies, Data Analysis and Applications, 8(2), 282-307.

Sharma, M. (2012), The Historical Simulation Method for Value-at-Risk: A Research Based Evaluation of the Industry Favorite. Available from: https://www.papers.ssrn.com/sol3/papers.cfm?abstract_id=2042594

So, M.K.P., Yu, P.L.H. (2006), Empirical analysis of GARCH models in value at risk estimation. Journal of International Financial Markets, Institutions and Money, 16(2), 180-197.

Storti, G., Wang, C. (2022), Nonparametric expected shortfall forecasting incorporating weighted quantiles. International Journal of Forecasting, 38(1), 224-239.

Taylor, J.W. (2019), Forecasting value at risk and expected shortfall using a semiparametric approach based on the asymmetric Laplace distribution. Journal of Business and Economic Statistics, 37(1), 121-133.

Thomson, D., Van Vuuren, G. (2018), Attribution of hedge fund returns using a Kalman filter. Applied Economics, 50(9), 1043-1058.

Tsay, R.S. (2010), Analysis of Financial Time Series. 3rd ed. Hoboken, New Jersey: John Wiley and Sons.

Van Rooyen, R., Van Vuuren, G. (2022), Tactical asset allocation using the Kalman filter. Investment Analysts Journal, 51(3), 202-215.

Zhang, Y., Nadarajah, S. (2018), A review of backtesting for value at risk. Communications in Statistics-Theory and Methods, 47(15), 3616-3639.

Žiković, S., Filer, R. (2009), Hybrid historical simulation VaR and ES: Performance in Developed and Emerging Markets. CESifo Working Paper Series (2820).